

# UNIQUENESS AND STABILITY OF SADDLE-SHAPED SOLUTIONS TO THE ALLEN-CAHN EQUATION

XAVIER CABRÉ

ABSTRACT. We establish the uniqueness of a saddle-shaped solution to the diffusion equation  $-\Delta u = f(u)$  in all of  $\mathbb{R}^{2m}$ , where  $f$  is of bistable type, in every even dimension  $2m \geq 2$ . In addition, we prove its stability whenever  $2m \geq 14$ .

Saddle-shaped solutions are odd with respect to the Simons cone  $\mathcal{C} = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^m : |x^1| = |x^2|\}$  and exist in all even dimensions. Their uniqueness was only known when  $2m = 2$ . On the other hand, they are known to be unstable in dimensions 2, 4, and 6. Their stability in dimensions 8, 10, and 12 remains an open question. In addition, since the Simons cone minimizes area when  $2m \geq 8$ , saddle-shaped solutions are expected to be global minimizers when  $2m \geq 8$ , or at least in higher dimensions. This is a property stronger than stability which is not yet established in any dimension.

## 1. INTRODUCTION

This paper concerns saddle-shaped solutions to bistable diffusion equations

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^{2m}, \quad (1.1)$$

where  $2m$  is an even integer. Throughout the paper we assume that  $f$  is a  $C^{2,\alpha}$  function on  $[-1, 1]$ , for some  $\alpha \in (0, 1)$ , such that

$$f \text{ is odd, } f(0) = f(1) = 0, \text{ and } f'' < 0 \text{ in } (0, 1). \quad (1.2)$$

As a consequence we have  $f > 0$  in  $(0, 1)$ . Under these assumptions, we say that  $f$  is of bistable type. A typical example is the nonlinearity  $f(u) = u - u^3$  in the Allen-Cahn equation.

For  $x = (x_1, \dots, x_{2m}) \in \mathbb{R}^{2m}$ , consider the two radial variables

$$\begin{cases} s &= (x_1^2 + \dots + x_m^2)^{1/2} \\ t &= (x_{m+1}^2 + \dots + x_{2m}^2)^{1/2}. \end{cases}$$

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The author was supported by grants MTM2008-06349-C03-01 (Spain) and 2009SGR-345 (Catalunya).

A *saddle-shaped solution* of (1.1) is a solution  $u$  of (1.1) which depends only on  $s$  and  $t$ , satisfies  $|u| < 1$ , is positive in  $\{s > t\}$ , and is odd with respect to  $\{s = t\}$ , i.e.,  $u(t, s) = -u(s, t)$ .

Consider also the variables

$$\begin{cases} y &= (s+t)/\sqrt{2} \\ z &= (s-t)/\sqrt{2}, \end{cases}$$

which satisfy  $y \geq 0$  and  $-y \leq z \leq y$ . An important set in what follows is the Simons cone (see Figure 1), defined by

$$\mathcal{C} = \{s = t\} = \{z = 0\} = \partial\mathcal{O},$$

where

$$\mathcal{O} = \{s > t\} = \{z > 0\} \subset \mathbb{R}^{2m}.$$

The cone  $\mathcal{C}$  has zero mean curvature at every  $x \in \mathcal{C} \setminus \{0\}$ , in every dimension  $2m \geq 2$ . However, it is only in dimensions  $2m \geq 8$  that  $\mathcal{C}$  is in addition a minimizer of the area functional. Furthermore,  $\mathcal{C}$  is stable only in these same dimensions; see [10] and references therein. We will see that similar properties hold, or are expected to hold, for saddle-shaped solutions of (1.1).

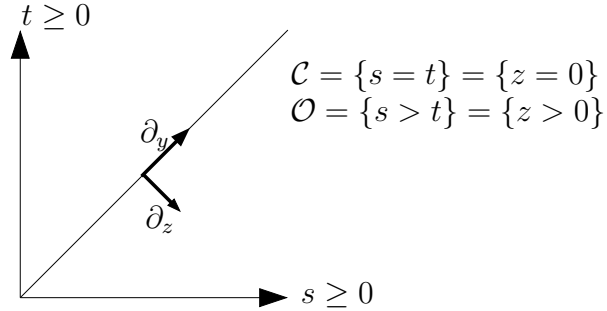


FIGURE 1. The Simons cone  $\mathcal{C}$ . The  $(s, t)$  and  $(y, z)$  variables

Under our assumptions (1.2) on  $f$ , there exists a unique increasing solution of  $-\Delta u = f(u)$  in all of  $\mathbb{R}$  up to translations of the independent variable; see, e.g., Lemma 4.3 of [9]. It has limits  $\pm 1$  at  $\pm\infty$ . We normalize it to vanish at the origin and we call it  $u_0$ . Thus, we have

$$\begin{cases} u_0 : \mathbb{R} \rightarrow (-1, 1), & -\ddot{u}_0 = f(u_0) \text{ in } \mathbb{R}, \\ u_0(0) = 0, \dot{u}_0 > 0 & \text{ in } \mathbb{R}, \text{ and } u_0(\tau) \rightarrow \pm 1 \text{ as } \tau \rightarrow \pm\infty. \end{cases}$$

For the Allen-Cahn nonlinearity  $f(u) = u - u^3$ , the solution  $u_0$  can be computed explicitly and it is given by  $u_0(\tau) = \tanh(\tau/\sqrt{2})$ .

It is simple to check that  $|z|$  is the distance in  $\mathbb{R}^{2m}$  from any point  $x \in \mathbb{R}^{2m}$  to the Simons cone  $\mathcal{C}$ ; see Lemma 4.2 of [9]. This is important when showing that the function

$$U(x) := u_0\left(\frac{s-t}{\sqrt{2}}\right) = u_0(z) \quad \text{for } x \in \mathbb{R}^{2m} \quad (1.3)$$

describes the asymptotic behavior of saddle-shaped solutions at infinity. This was established by J. Terra and the author [10] and it is stated in Theorem 1.1 below. Note that  $U$  is a Lipschitz function in  $\mathbb{R}^{2m}$ , but it is not differentiable at  $\{st = 0\}$ . Using (1.15), which is equation (1.1) expressed in the  $(s, t)$  variables, one sees that  $-\Delta U > f(U)$  in  $\{s > t > 0\}$ , i.e.,  $U$  is a strict supersolution in  $\{s > t > 0\}$  —a fact that we will not use in this paper.

The energy functional associated to equation (1.1) is

$$\mathcal{E}(u, \Omega) := \int_{\Omega} \left\{ \frac{1}{2} |\nabla u|^2 + G(u) \right\} dx, \quad \text{where } G' = -f. \quad (1.4)$$

We say that a bounded solution  $u$  of (1.1) is *stable* if the second variation of energy  $\delta^2 \mathcal{E} / \delta^2 \xi$  with respect to compactly supported smooth perturbations  $\xi$  is nonnegative. That is, if

$$Q_u(\xi) := \int_{\mathbb{R}^{2m}} \{ |\nabla \xi|^2 - f'(u) \xi^2 \} dx \geq 0 \quad \text{for all } \xi \in C_c^\infty(\mathbb{R}^{2m}). \quad (1.5)$$

We say that  $u$  is *unstable* when  $u$  is not stable. The stability of  $u$  is equivalent to requiring the linearized operator  $-\Delta - f'(u)$  to have positive first eigenvalue (or to satisfy the maximum principle) in every smooth bounded domain of  $\mathbb{R}^{2m}$ ; see section 2 for these questions.

A bounded function  $u \in C^1(\mathbb{R}^{2m})$  is said to be a *global minimizer* of (1.1) when  $\mathcal{E}(u, \Omega) \leq \mathcal{E}(v, \Omega)$  for every bounded domain  $\Omega$  and function  $v \in C^1(\overline{\Omega})$  such that  $v \equiv u$  on  $\partial\Omega$ . Clearly, every global minimizer is a stable solution.

The following results were proven in [9, 10] by J. Terra and the author.

**Theorem 1.1 (Cabr -Terra [9, 10]).** *Assume that  $f$  satisfies (1.2).*

(a) *For every even dimension  $2m \geq 2$ , there exists a saddle-shaped solution  $u \in C^2(\mathbb{R}^{2m})$  of  $-\Delta u = f(u)$  in  $\mathbb{R}^{2m}$ , that is, a solution  $u$  of this equation with  $|u| < 1$  and such that*

- *$u$  depends only on the variables  $s$  and  $t$ . We write  $u = u(s, t)$ ;*
- *$u > 0$  in  $\mathcal{O} = \{s > t\}$ ;*
- *$u(t, s) = -u(s, t)$  in  $\mathbb{R}^{2m}$ .*

(b) For  $2m \geq 2$ , every saddle-shaped solution  $u$  satisfies

$$\| |u - U| + |\nabla(u - U)| \|_{L^\infty(\mathbb{R}^{2m} \setminus B_R(0))} \rightarrow 0 \quad \text{as } R \rightarrow \infty, \quad (1.6)$$

where  $U$  is defined by (1.3).

(c) When  $2 \leq 2m \leq 6$ , every saddle-shaped solution  $u$  is unstable. Furthermore, in dimensions 4 and 6 every saddle-shaped solution has infinite Morse index in the sense of Definition 1.8 of [10].

Part (a) of the theorem concerns existence. It can be established in different ways, all of them rather simple—for instance, variationally as in section 3 of [9], or through monotone iteration as in section 3 of [10].

Part (b) gives the asymptotic behavior of saddle-shaped solutions at infinity. In this paper we will also need the asymptotics of second derivatives. See Lemma 4.1 below, where we sketch also the proof of (1.6).

The uniqueness of a saddle-shaped solution was only known in dimension 2, even for the Allen-Cahn nonlinearity. This was established by Dang, Fife, and Peletier [11], who also proved—in dimension 2—existence of a saddle-shaped solution and some results on its asymptotic behavior and its monotonicity properties.

The instability of the saddle-shaped solution in  $\mathbb{R}^2$ , indicated in a partial result of [11], was studied in detail by Schatzman [16] by analyzing the linearized operator at the saddle-shaped solution and showing that, when  $f(u) = u - u^3$ , this operator has exactly one negative eigenvalue. That is, the saddle-shaped solution of the Allen-Cahn equation in  $\mathbb{R}^2$  has Morse index 1—in contrast with our result Theorem 1.1(c) in dimensions 4 and 6. See [10] for more comments on this.

Our first main result is the following uniqueness theorem.

**Theorem 1.2.** *Assume that  $f$  satisfies (1.2). Then, for every even dimension  $2m \geq 2$ , there exists a unique saddle-shaped solution  $u$  of  $-\Delta u = f(u)$  in  $\mathbb{R}^{2m}$ .*

The uniqueness result will follow (see section 3) from two main ingredients: the asymptotics (1.6) at infinity for saddle-shaped solutions and the following new result—a maximum principle in  $\mathcal{O} = \{s > t\}$  for the linearized operator at every saddle-shaped solution.

**Proposition 1.3.** *Assume that  $f$  satisfies (1.2). Let  $u$  be a saddle-shaped solution of (1.1), where  $2m \geq 2$ .*

*Then, the maximum principle holds for the operator*

$$L_u := \Delta + f'(u(x)) \quad \text{in } \mathcal{O} = \{s > t\},$$

in the sense that whenever  $v \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$  satisfies

$$L_u v \geq 0 \text{ in } \mathcal{O}, \quad v \leq 0 \text{ on } \partial\mathcal{O}, \quad \text{and} \quad \limsup_{x \in \mathcal{O}, |x| \rightarrow \infty} v(x) \leq 0, \quad (1.7)$$

then necessarily  $v \leq 0$  in  $\mathcal{O}$ .

We establish this result in section 2 using as key ingredient a maximum principle in the “narrow” domain  $\{t < s < t + \varepsilon\}$ , where  $\varepsilon$  is small; see Lemma 2.3 below.

As we explain at the end of this introduction, due to a connection between minimal surfaces and solutions of the Allen-Cahn equation, saddle-shaped solutions are expected to be global minimizers of (1.1) (as defined in the beginning of this introduction) in dimensions 8 and higher—or at least in dimensions high enough. This is still an open question—already raised in 2002 by Jerison and Monneau; see conjecture C4 and section 1.3 in [13].

Towards the understanding of this minimality property, in this article we establish the stability of saddle-shaped solutions in dimensions 14 and higher. This is our second main result. Of course, stability is a weaker property than minimality. Stability in dimensions 8, 10, and 12 remains an open question.

To state the result, we need to choose a number  $b$  such that

$$b(b - m + 2) + m - 1 \leq 0. \quad (1.8)$$

This is possible only when  $m \geq 7$ , in which case one can take any

$$b \in [b_-, b_+], \quad \text{where } b_{\pm} = \frac{m-2}{2} \pm \frac{\sqrt{(m-2)^2 - 4(m-1)}}{2}. \quad (1.9)$$

It follows that  $b > 0$ .

**Theorem 1.4.** *Assume that  $f$  satisfies (1.2). If  $2m \geq 14$ , the saddle-shaped solution  $u$  of (1.1) is stable in  $\mathbb{R}^{2m}$ , i.e., (1.5) holds. Furthermore, for every  $b > 0$  satisfying (1.9), the function*

$$\varphi := t^{-b}u_s - s^{-b}u_t \quad (1.10)$$

*is of class  $C^2$  and positive in  $\mathbb{R}^{2m} \setminus \{st = 0\}$ , it is even with respect to the Simons cone, and it is a supersolution of the linearized equation*

$$\Delta\varphi + f'(u)\varphi \leq 0 \quad \text{in } \mathbb{R}^{2m} \setminus \{st = 0\}.$$

The statement on the stability of the saddle-shaped solution will follow immediately from the properties of  $\varphi$  stated in the theorem. The key ingredients to establish Theorem 1.4 are the following monotonicity and convexity properties of saddle-shaped solutions. They hold in every even dimension.

**Proposition 1.5.** *Assume that  $f$  satisfies (1.2). Let  $u$  be the saddle-shaped solution of (1.1), where  $2m \geq 2$ . We then have*

$$u_y > 0 \quad \text{in } \mathcal{O} = \{s > t\}, \quad (1.11)$$

$$-u_t > 0 \quad \text{in } \mathcal{O} \setminus \{t = 0\} = \{s > t > 0\}, \quad (1.12)$$

and

$$u_{st} > 0 \quad \text{in } \mathcal{O} \setminus \{t = 0\} = \{s > t > 0\}. \quad (1.13)$$

The cone of monotonicity generated by  $\partial_y$  and  $-\partial_t$  is the optimal one (i.e., the largest one) holding at all points of  $\{s > t > 0\}$ . In Figure 2 we draw this cone, and also the shape of the level sets of a saddle-shaped solution —where we take into account the asymptotic result (1.6).

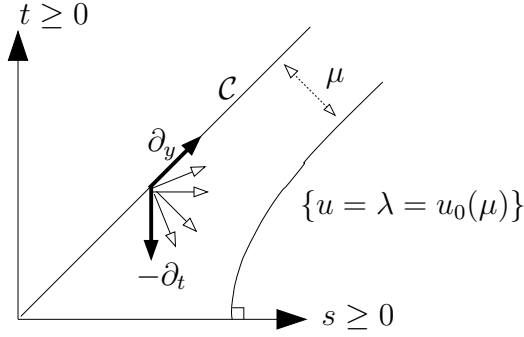


FIGURE 2. The cone of monotonicity and the level sets of the saddle-shaped solution

The monotonicity properties (1.11) and (1.12) were first proven by J. Terra and the author in [10] for the so-called maximal saddle-shaped solution —at that point uniqueness of saddle-shaped solution was not known. Here we establish these properties using a different method.

The second derivative property (1.13) is a new fact proved in this paper. It is a crucial ingredient to establish stability in dimensions 14 and higher.

The three inequalities in Proposition 1.5 will be proven using our maximum principle for the linearized operator in  $\mathcal{O}$ , or rather a slightly more general version: Proposition 2.2 of next section.

To establish such maximum principle, note that since  $f(0) = 0$  and  $f'' < 0$  in  $(0, 1)$ , we have  $f(\rho)/\rho > f'(\rho)$  for all  $0 < \rho < 1$ . Thus, if  $u$  is a saddle-shaped solution then

$$-\Delta u = f(u) > f'(u)u \quad \text{in } \mathcal{O}. \quad (1.14)$$

That is,  $u$  is a positive and strict supersolution of the linearized operator  $\Delta + f'(u)$  at  $u$  in all  $\mathcal{O}$ . This will be one of the ingredients (but not the only one since  $\inf_{\mathcal{O}} u = 0$ ) to establish the maximum principle in  $\mathcal{O}$  for the linearized operator. The other ingredient will be a maximum principle for the linearized operator in the “narrow” domain  $\{t < s < t + \varepsilon\}$ , Lemma 2.3.

To prove Proposition 1.5, we will apply the maximum principle of Proposition 2.2 to the equations satisfied by  $u_s$ ,  $u_t$ ,  $u_y$ , and  $u_{st}$ . For this, note that equation (1.1) in the  $(s, t)$  variables reads

$$u_{ss} + u_{tt} + (m-1)\left(\frac{u_s}{s} + \frac{u_t}{t}\right) + f(u) = 0 \quad (1.15)$$

for  $s > 0$  and  $t > 0$ , while in the  $(y, z)$  variables it becomes

$$u_{yy} + u_{zz} + \frac{2(m-1)}{y^2 - z^2}(yu_y - zu_z) + f(u) = 0 \quad (1.16)$$

for  $|z| < y$ .

Finally, let us explain why  $2m \leq 6$  is relevant in Theorem 1.1(c) and why stability and minimality are expected in higher dimensions. Due to a relation between the Allen-Cahn equation and the theory of minimal surfaces (see [15, 13, 12, 1]), every level set of a global minimizer of (1.1) should converge at infinity in some weak sense to the boundary of a minimal set —minimal in the variational sense, that is, minimizing perimeter. Note now that the zero level set of a saddle-shaped solution is the Simons cone  $\mathcal{C}$ . It is easy to verify that  $\mathcal{C}$  has zero mean curvature at every  $x \in \mathcal{C} \setminus \{0\}$ , in every dimension  $2m \geq 2$ . However, it is only in dimensions  $2m \geq 8$  when  $\mathcal{C}$  is in addition a minimizer of the area functional, i.e., it is a minimal cone in the variational sense. Furthermore,  $\mathcal{C}$  is stable only in these same dimensions. See [10] and references therein for these questions.

Furthermore, a deep theorem states that the boundary of a minimal set in all of  $\mathbb{R}^n$  must be a hyperplane if  $n \leq 7$ . Instead, in  $\mathbb{R}^8$  and higher dimensions, there exist minimal sets different than half-spaces —the simplest example being the Simons cone.

The analogue for the Allen-Cahn equation of the first of these two results is well understood. Indeed, a deep theorem of Savin [15] states that in dimensions  $n \leq 7$ , 1D solutions (i.e., solutions depending only on one Euclidean variable) are the only global minimizers of the Allen-Cahn equation. Note that this result makes no assumption on the monotonicity or limits at infinity of the solution. That is:

**Theorem 1.6** (Savin [15]). *Assume that  $n \leq 7$  and that  $u$  is a global minimizer of  $-\Delta u = u - u^3$  in  $\mathbb{R}^n$ . Then, the level sets of  $u$  are hyperplanes.*

However, the analogue for the Allen-Cahn equation of the second statement (i.e., the minimality of the Simons cone when  $2m \geq 8$ ) is not yet understood. That is, the possible minimality in  $\mathbb{R}^8$  (or at least in higher dimensions) of the saddle-shaped solution is still unknown. This question was already raised in 2002 by Jerison and Monneau [13].

**Open Question 1.7.** Is the saddle-shaped solution a global minimizer of the Allen-Cahn equation in  $\mathbb{R}^{2m}$  for  $2m \geq 8$ , or at least in higher even dimensions?

Related to this, in  $\mathbb{R}^9$  it is known the existence of a global minimizer to the Allen-Cahn equation which is not 1D (i.e., with level sets different than hyperplanes). This is the solution in  $\mathbb{R}^9$ , monotone in the  $x_9$  variable, constructed by del Pino, Kowalczyk, and Wei [12]. Since this monotone solution is known to have limits  $\pm 1$  as  $x_9 \rightarrow \pm\infty$ , a result of Alberti, Ambrosio, and the author [1] guarantees that the solution is indeed a global minimizer.

In this direction, a positive answer to the Open Question 1.7 above would give an alternative way to that of [12] to prove the existence of a counter-example to the conjecture of De Giorgi in  $\mathbb{R}^9$ . Indeed, saddle-shaped solutions are even functions of each coordinate  $x_i$ . Thus, by a result of Jerison and Monneau [13], if the saddle-shaped solution were a global minimizer in  $\mathbb{R}^{2m}$ , then the conjecture of De Giorgi on monotone solutions would not hold in  $\mathbb{R}^{2m+1}$ . Indeed, from the 1D solution depending only on  $x_{2m+1}$  and from a global minimizing saddle-shaped solution depending only on  $(x_1, \dots, x_{2m})$ , [13] constructs in a natural way a solution in  $\mathbb{R}^{2m+1}$  which is monotone in the last variable  $x_{2m+1}$  and which is not 1D. However, it is not proved that the solution of [13] has limits  $\pm 1$  as  $x_{2m+1} \rightarrow \pm\infty$  —a property known for the solution in  $\mathbb{R}^9$  of [12].

The plan of the paper is the following. In section 2 we prove the maximum principle for the linearized operator in  $\mathcal{O}$ . Section 3 establishes the uniqueness of saddle-shaped solution, Theorem 1.2. In section 4 we establish Proposition 1.5 on the sign of derivatives of  $u$ . Finally, section 5 concerns the supersolution of the linearized equation and completes the proof of our stability result, Theorem 1.4.



2. THE MAXIMUM PRINCIPLE FOR  $\Delta + f'(u)$  IN  $\mathcal{O}$ 

Let us consider linear operators of the form

$$Lv := \Delta v + c(x)v,$$

where  $c$  is a continuous function (perhaps unbounded) in an open set  $\Omega$  of  $\mathbb{R}^n$ , i.e.,  $c \in C(\Omega)$ .

**Definition 2.1.** Let  $\Omega$  be an open set of  $\mathbb{R}^n$ . We say that the maximum principle holds for the operator  $L$  in  $\Omega$  if, whenever  $v \in C^2(\Omega) \cap C(\overline{\Omega})$  satisfies

$$Lv \geq 0 \text{ in } \Omega, \quad v \leq 0 \text{ on } \partial\Omega, \quad \text{and} \quad \limsup_{x \in \Omega, |x| \rightarrow \infty} v(x) \leq 0, \quad (2.1)$$

then necessarily  $v \leq 0$  in  $\Omega$ .

The last condition in (2.1) only plays a role when  $\Omega$  is unbounded.

The main result in this section is the following.

**Proposition 2.2.** *Assume that  $f$  satisfies (1.2). Let  $u$  be a saddle-shaped solution of (1.1), where  $2m \geq 2$ . Let  $\Omega \subset \mathcal{O} = \{s > t\}$  be an open set and  $c \in C(\Omega)$  with  $c \leq 0$  in  $\Omega$ .*

*Then, the maximum principle holds for the operator  $L_u + c(x) = \Delta + \{f'(u(x)) + c(x)\}$  in  $\Omega$ .*

We will use this result to prove both Theorem 1.2 on uniqueness and Proposition 1.5 on the sign of  $u_y$ ,  $u_t$ , and  $u_{st}$ . For this we will use the previous proposition both with  $\Omega = \mathcal{O} = \{s > t\}$  and with  $\Omega = \mathcal{O} \setminus \{t = 0\} = \{s > t > 0\}$ . We will need to use it with different choices of coefficient  $c = c(x)$ , with  $c$  continuous and nonpositive in  $\Omega$  but unbounded below.

The rest of this section is devoted to prove Proposition 2.2. For this, recall that the typical way towards establishing the maximum principle for an operator  $L$  in an open set  $\Omega$  is to first show that

$$\text{“there exists a positive supersolution } \phi \text{ of } L\phi = 0 \text{ in } \Omega\text{”}. \quad (2.2)$$

If this holds, then adding one of various additional assumptions on  $\phi$  (the simplest one being  $\phi \geq c > 0$  with  $c$  a positive constant), it does guarantee the maximum principle to hold; see [5]. Indeed, in bounded domains, (2.2) is a necessary —and “almost” sufficient— condition for the maximum principle to hold; see Corollary 2.1 of [5]. However, in unbounded domains one has to be more careful to deal with infinity.

Recall that when  $u$  is a saddle-shaped solution of (1.1),  $u$  is a positive supersolution of the linearized operator  $\Delta + f'(u)$  at  $u$  in all  $\mathcal{O}$ ; see

(1.14). However,  $\inf_{\mathcal{O}} u = 0$  since  $u = 0$  on  $\partial\mathcal{O}$ . Nevertheless, we claim that for every given  $\varepsilon > 0$ , we have that

$$u \geq \delta > 0 \quad \text{in } \mathcal{O}_\varepsilon := \{s > t + \varepsilon\} \quad (2.3)$$

for some positive constant  $\delta$  (which may depend on the particular solution  $u$ ). Indeed,  $U(x) = u_0(z) \geq u_0(\varepsilon/\sqrt{2}) > 0$  in  $\mathcal{O}_\varepsilon$ . Hence, by the asymptotic behavior of saddle-shaped solutions at infinity, Theorem 1.1(b), there exists a radius  $R > 0$  such that  $u(x) \geq u_0(\varepsilon/\sqrt{2})/2$  (a positive constant) if  $|x| > R$  and  $x \in \mathcal{O}_\varepsilon$ . Now, since  $u$  is positive in the compact set  $\overline{\mathcal{O}_\varepsilon} \cap \overline{B_R(0)}$ , we conclude the claim.

The lower bound (2.3) in  $\mathcal{O}_\varepsilon$  together with the following maximum principle in  $\mathcal{N}_\varepsilon := \mathcal{O} \setminus \overline{\mathcal{O}_\varepsilon}$ , a “narrow” domain in a sense explained later, will lead to the maximum principle for  $L_u$  in all of  $\mathcal{O}$ .

**Lemma 2.3.** *Let  $2m \geq 2$ ,  $\varepsilon > 0$ , and*

$$\mathcal{N}_\varepsilon := \{t < s < t + \varepsilon\} \subset \mathbb{R}^{2m}.$$

*Let  $H \subset \mathcal{N}_\varepsilon$  be an open set and  $\tilde{c} \in C(H)$  satisfy  $\tilde{c}_+ \in L^\infty(H)$ , where  $\tilde{c}_+$  denotes the positive part of  $\tilde{c}$ .*

*Then, the maximum principle holds for the operator  $\Delta + \tilde{c}(x)$  in  $H$  whenever*

$$C_m \varepsilon^2 \|\tilde{c}_+\|_{L^\infty(H)} < 1, \quad (2.4)$$

*where  $C_m$  is a positive constant depending only on  $m$ .*

The constant  $C_m$  can be taken to be equal to 3 for all  $m$  —this will be seen below, in our second proof of the lemma. Note that in this result  $\tilde{c}$  is allowed to change sign —in contrast with Proposition 2.2.

Using Lemma 2.3 we can now prove Proposition 2.2.

*Proof of Proposition 2.2.* Let  $u$  be a saddle-shaped solution of (1.1) and let

$$Lv := L_u v + c(x)v = \Delta v + \{f'(u(x)) + c(x)\}v.$$

Since  $c \leq 0$  in  $\Omega \subset \mathcal{O}$ , then  $f'(u) + c \leq f'(u) \leq \max_{[0,1]} f' = f'(0)$  in  $\Omega$ . Choosing  $\varepsilon = (2C_m f'(0))^{-1/2}$ , Lemma 2.3 states that the maximum principle holds for the operator  $L$  in any open subset of  $\mathcal{N}_\varepsilon$ .

Let

$$\Omega_\varepsilon := \Omega \cap \mathcal{O}_\varepsilon = \Omega \cap \{s > t + \varepsilon\}$$

and

$$H_\varepsilon := \Omega \cap \mathcal{N}_\varepsilon = \Omega \cap \{t < s < t + \varepsilon\} \subset \mathcal{N}_\varepsilon.$$

Note that

$$\partial\Omega_\varepsilon \subset \partial\Omega \cup (\Omega \cap \{s = t + \varepsilon\}), \quad (2.5)$$

$$\partial H_\varepsilon \subset \partial\Omega \cup (\Omega \cap \{s = t + \varepsilon\}) \subset \partial\Omega \cup \overline{\Omega}_\varepsilon \quad (2.6)$$

since  $\Omega \cap \{s = t\} = \emptyset$ , and

$$\overline{\Omega} = \overline{\Omega}_\varepsilon \cup \overline{H}_\varepsilon. \quad (2.7)$$

Recall that  $u > 0$  in  $\Omega \subset \mathcal{O}$  and that by (2.3) we know that

$$u \geq \delta > 0 \quad \text{in } \Omega_\varepsilon \quad (2.8)$$

for some constant  $\delta > 0$ . In addition, by (1.14),

$$Lu = \Delta u + \{f'(u) + c\}u \leq \Delta u + f'(u)u < 0 \quad \text{in } \Omega. \quad (2.9)$$

Let  $v \in C^2(\Omega) \cap C(\overline{\Omega})$ , as in the definition of the maximum principle, satisfy

$$Lv \geq 0 \text{ in } \Omega, \quad v \leq 0 \text{ on } \partial\Omega, \quad \text{and} \quad \limsup_{x \in \Omega, |x| \rightarrow \infty} v(x) \leq 0. \quad (2.10)$$

Consider

$$w := \frac{v}{u} \quad \text{in } \Omega.$$

By (2.8) and the hypotheses (2.10) on  $v \in C(\overline{\Omega})$ ,  $w$  is bounded above in  $\Omega_\varepsilon$ .

Assume that

$$S := \sup_{\overline{\Omega}_\varepsilon} w > 0. \quad (2.11)$$

Then, by the two last conditions in (2.10) and by (2.5), this supremum must be achieved at a point  $x_0 \in \Omega_\varepsilon \cup (\Omega \cap \{s = t + \varepsilon\}) \subset \Omega$ .

We have that  $v - Su \leq 0$  in  $\overline{\Omega}_\varepsilon$ . Therefore,  $v - Su$  is a subsolution for  $L$  in  $H_\varepsilon$  and nonpositive on  $\partial H_\varepsilon$ , by (2.6), and at infinity. Thus, the maximum principle in  $H_\varepsilon$ , Lemma 2.3, leads to  $v - Su \leq 0$  in  $H_\varepsilon$  and hence, by (2.7), also

$$v - Su \leq 0 \quad \text{in } \overline{\Omega}.$$

We deduce that  $S = w(x_0) = \sup_{\overline{\Omega}_\varepsilon} w = \sup_{\overline{\Omega}} w$ , and thus the point  $x_0 \in \Omega$  obtained before is an interior maximum of  $w$ . Now, note that

$$\begin{aligned} \operatorname{div}(u^2 \nabla w) &= \operatorname{div}(\nabla v u - v \nabla u) = \Delta v u - v \Delta u = Lv u - v Lu \\ &\geq -v Lu \quad \text{in } \Omega. \end{aligned}$$

Hence

$$\Delta w + 2u^{-1} \nabla u \nabla w + u^{-1} Lu w \geq 0 \quad \text{in } \Omega. \quad (2.12)$$

But at the interior point  $x_0 \in \Omega$  of maximum of  $w$ , we have

$$\begin{aligned} (\Delta w + 2u^{-1} \nabla u \nabla w + u^{-1} Lu w)(x_0) &\leq \\ &\leq (u^{-1} Lu w)(x_0) = Su^{-1}(x_0) Lu(x_0) < 0 \end{aligned}$$

by (2.9), a contradiction with (2.12). Thus, (2.11) does not hold. We conclude  $\sup_{\overline{\Omega}_\varepsilon} w \leq 0$  and hence  $v \leq 0$  in  $\overline{\Omega}_\varepsilon$ .

Finally, arguing for  $v$  exactly as done before for  $v - Su$ , we deduce that  $v \leq 0$  on  $\partial H_\varepsilon$ . Then, the maximum principle in  $H_\varepsilon$  leads to  $v \leq 0$  in  $H_\varepsilon$ , and thus also in all  $\Omega$  by (2.7).  $\square$

The domain  $\mathcal{N}_\varepsilon = \{t < s < t + \varepsilon\}$  is a “narrow” domain in the sense of [5], and thus Lemma 2.3 follows from a very general maximum principle in “narrow” domains due to Berestycki, Nirenberg, and Varadhan [5]. However, for completeness, below we give two different simple proofs of the lemma. First, let us explain what “narrow” means and why the lemma follows from results of [5, 8].

Let  $H \subset \mathcal{N}_\varepsilon \subset \{t < s < t + \varepsilon\}$  be an open set, as in Lemma 2.3. Let  $x$  be any point in  $H$ . It is simple to check that the distance from  $x$  to the Simons cone  $\mathcal{C}$  is given by the  $z$  coordinate of  $x$ , i.e., by  $(s_x - t_x)/\sqrt{2}$ ; see Lemma 4.2 in [9]. Hence, there exists a point  $\bar{x} \in \mathcal{C}$  such that  $|x - \bar{x}| = (s_x - t_x)/\sqrt{2} < \varepsilon/\sqrt{2} < (3/4)\varepsilon$ . Thus

$$B_{\varepsilon/4}(\bar{x}) \setminus \mathcal{O} \subset B_{\varepsilon/4}(\bar{x}) \setminus H \subset B_\varepsilon(x) \setminus H,$$

and hence, since  $\bar{x} \in \mathcal{C}$ ,

$$\begin{aligned} 2^{-1-4m}|B_\varepsilon(x)| &= (1/2)|B_{\varepsilon/4}(\bar{x})| = |B_{\varepsilon/4}(\bar{x}) \setminus \mathcal{O}| \\ &\leq |B_\varepsilon(x) \setminus H|. \end{aligned} \tag{2.13}$$

Lemma 2.3 now follows from Definitions 2.2 and 5.1 and Theorem 5.2(i) of [8] —a slightly more general version than the result of [5] to include unbounded domains. The specific dependence (2.4) is the same as the one obtained in the proof of Theorem 5.2(i) of [8].

Nevertheless, for completeness we present next two proofs of Lemma 2.3. The first one follows the proof in [8] and it was found by the author in [6]. Replacing its technical tools (the mean value inequality for superharmonic functions used below by the Krylov-Safonov weak Harnack inequality), it applies to general “narrow” domains and to operators in non-divergence form with bounded measurable coefficients. Instead, our second proof will use strongly the specific “shape” of the domain  $\mathcal{N}_\varepsilon$ .

*First proof of Lemma 2.3.* Let  $H \subset \mathcal{N}_\varepsilon \subset \{t < s < t + \varepsilon\}$  be an open set and  $v \in C^2(H) \cap C(\overline{H})$  satisfy

$$Lv := \Delta v + \tilde{c}(x)v \geq 0 \text{ in } H, \quad v \leq 0 \text{ on } \partial H, \quad \text{and} \quad \limsup_{x \in H, |x| \rightarrow \infty} v(x) \leq 0.$$

Arguing by contradiction, assume that  $\sup_H v > 0$ . It follows that the supremum of  $v$  is achieved at some point  $x_0 \in H$ :

$$\sup_H v = v(x_0) > 0.$$

Let

$$K := \|\tilde{c}_+\|_{L^\infty(H)}$$

and

$$\phi(x) := (4m)^{-1}Kv(x_0)(\varepsilon^2 - |x - x_0|^2) \quad \text{for } x \in \mathbb{R}^{2m}.$$

Consider now the open set  $H \cap \{v > 0\}$ . We have

$$-\Delta v \leq \tilde{c}v \leq \|\tilde{c}_+\|_{L^\infty(H)}v = Kv \leq Kv(x_0) = -\Delta\phi \quad \text{in } H \cap \{v > 0\}.$$

Thus,  $v - \phi$  is subharmonic in  $B_\varepsilon(x_0) \cap (H \cap \{v > 0\})$ . In addition, on  $B_\varepsilon(x_0) \cap \partial(H \cap \{v > 0\})$  we have  $v - \phi \leq v \leq 0$ . Thus, its positive part  $(v - \phi)_+$ , extended to be zero in  $B_\varepsilon(x_0) \setminus (H \cap \{v > 0\})$ , is a continuous function which is subharmonic in the viscosity sense (or in the distributional sense) in  $B_\varepsilon(x_0)$ .

We apply to  $w := v(x_0) - (v - \phi)_+$  the mean value inequality in the ball  $B_\varepsilon(x_0)$  for superharmonic functions in the viscosity (or distributional) sense. Note also that  $w > 0$  in  $B_\varepsilon(x_0)$  and recall the “narrowness” condition (2.13) with  $x = x_0 \in H$ . We have

$$\begin{aligned} 2^{-1-4m}v(x_0) &\leq \frac{|B_\varepsilon(x_0) \setminus (H \cap \{v > 0\})|}{|B_\varepsilon(x_0)|}v(x_0) \\ &= \frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0) \setminus (H \cap \{v > 0\})} w \\ &\leq \frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} w \leq w(x_0) \\ &= v(x_0) - (v(x_0) - \phi(x_0))_+ \\ &\leq \phi(x_0) = (4m)^{-1}\varepsilon^2 Kv(x_0). \end{aligned}$$

Thus, since we assumed  $\sup_H v = v(x_0) > 0$ , we get a contradiction whenever  $(4m)^{-1}\varepsilon^2 K < 2^{-1-4m}$ .  $\square$

The rest of this section is devoted to give another simple proof of Lemma 2.3. In contrast with the previous one, the following proof is based on the specific form of the domain  $\mathcal{N}_\varepsilon$ .

To give the proof, we first need to establish the following easy result.

**Lemma 2.4.** *Let  $H$  be an open set of  $\mathbb{R}^n$ ,  $\tilde{c} \in C(H)$ , and  $L = \Delta + \tilde{c}(x)$ . Assume that there exists a function  $\phi \in C(\overline{H})$  (not necessarily bounded above) such that*

$$\phi \geq \delta > 0 \quad \text{in } \overline{H}$$

*for some constant  $\delta$ . Assume also that there exists an open set  $A \subset H$  such that  $\phi \in C^2(A)$ ,*

$$L\phi < 0 \quad \text{in } A, \tag{2.14}$$

and

$$\liminf_{\xi \rightarrow 0} \frac{\phi(x_0 + \xi) + \phi(x_0 - \xi) - 2\phi(x_0)}{|\xi|^2} = -\infty \quad \text{for all } x_0 \in H \setminus A. \quad (2.15)$$

Then, the maximum principle holds for  $L$  in  $H$ .

Even if it could be relaxed, note the strict inequality in (2.14).

Condition (2.15) prevents the function  $\phi$  to be “touched by below” at the point  $x_0$  by a  $C^2$  function (see the proof of the lemma for details). As an example, the function  $\phi(x) = -|x|$  satisfies (2.15) at  $x_0 = 0$ . Another example appearing in applications is the distance function to a given point  $p$  in a Riemannian manifold; it satisfies (2.15) at points  $x_0$  in the cut locus of  $p$  (see [7]). It also occurs with the distance to the boundary  $\partial H$  in an open set  $H$  of  $\mathbb{R}^n$  at a cut point  $x_0$  in  $H$  (see [14]).

*Proof of Lemma 2.4.* Let  $H \subset \mathbb{R}^n$  be an open set and  $v \in C^2(H) \cap C(\overline{H})$  satisfy

$$Lv = \Delta v + \tilde{c}(x)v \geq 0 \text{ in } H, \quad v \leq 0 \text{ on } \partial H, \quad \text{and} \quad \limsup_{x \in H, |x| \rightarrow \infty} v(x) \leq 0.$$

Consider the function

$$w := \frac{v}{\phi},$$

with  $\phi$  as in Lemma 2.4. We have that  $w$  is a continuous function in  $\overline{H}$  satisfying  $w \leq 0$  on  $\partial H$  and  $\limsup_{x \in H, |x| \rightarrow \infty} w(x) \leq 0$ . Thus,  $w$  is bounded above.

Arguing by contradiction, assume that  $S := \sup_H w > 0$ . This supremum will be achieved at some point  $x_0 \in H$ , by the nonpositiveness of the limsup of  $w$  at infinity.

We claim that  $x_0 \in A$ . Indeed, we have that

$$v \leq S\phi \quad \text{in } A \quad \text{and} \quad v(x_0) = S\phi(x_0).$$

It follows that the liminf for  $\phi$  in (2.15) is greater than or equal to the same liminf for  $S^{-1}v$ , which is finite since  $v \in C^2(H)$ . By (2.15), we conclude that  $x_0 \in A$ .

Now,  $v$ ,  $\phi$ , and  $w$  are  $C^2$  in  $A$  and we have

$$\begin{aligned} \operatorname{div}(\phi^2 \nabla w) &= \operatorname{div}(\nabla v \phi - v \nabla \phi) = \Delta v \phi - v \Delta \phi = Lv \phi - v L\phi \\ &\geq -v L\phi. \end{aligned}$$

Hence

$$\Delta w + 2\phi^{-1} \nabla \phi \nabla w + \phi^{-1} L\phi w \geq 0 \quad \text{in } A. \quad (2.16)$$

But at the point  $x_0 \in A$  of maximum of  $w$ , we have

$$\begin{aligned} (\Delta w + 2\phi^{-1}\nabla\phi\nabla w + \phi^{-1}L\phi w)(x_0) &\leq \\ &\leq (\phi^{-1}L\phi w)(x_0) = S\phi^{-1}(x_0)L\phi(x_0) < 0 \end{aligned}$$

by (2.14), a contradiction with (2.16).

Thus,  $\sup_H w \leq 0$  and hence  $v \leq 0$  in  $H$ .  $\square$

We can now give the second proof of the maximum principle in  $\mathcal{N}_\varepsilon$ .

*Second proof of Lemma 2.3.* Assume that

$$3\varepsilon^2\|\tilde{c}_+\|_{L^\infty(H)} < 1.$$

We apply Lemma 2.4 with the choice

$$\begin{aligned} \phi(x) = \phi(z) &:= (z + \varepsilon)(3\varepsilon - z) = 3\varepsilon^2 + 2\varepsilon z - z^2 \\ &= 3\varepsilon^2 + \frac{2\varepsilon}{\sqrt{2}}(s - t) - \frac{s^2 + t^2 - 2st}{2}. \end{aligned} \quad (2.17)$$

Note that  $0 < z < \varepsilon/\sqrt{2} < \varepsilon$  in  $\mathcal{N}_\varepsilon$ , and thus

$$2\varepsilon^2 \leq \phi \leq 6\varepsilon^2 \quad \text{in } \mathcal{N}_\varepsilon.$$

For the set  $A$  in Lemma 2.4 we choose

$$A = H \cap \{0 < t < s < t + \varepsilon\},$$

and thus

$$H \setminus A \subset \{t = 0 \text{ and } 0 < s < \varepsilon\}.$$

Given a point  $x_0 \in H \setminus A$ , since  $x_0$  is a point with the  $t$  coordinate  $t_0 = 0$  and with the  $s$  coordinate  $0 < s_0 < \varepsilon$ , (2.17) shows that in a neighborhood of  $x_0$  the function  $\phi$  is equal to a smooth function plus

$$(-\sqrt{2}\varepsilon + s)t.$$

Since  $-\sqrt{2}\varepsilon + s_0 < -\sqrt{2}\varepsilon + \varepsilon < 0$ , considering second order incremental quotients in the  $t$  variable, we see that the liminf in (2.15) for this function at the point  $x_0$  is equal to  $-\infty$ . Thus, the same holds for  $\phi$ .

Next, we have that  $\phi \in C^2(A)$  and, in  $A$ ,  $\phi_z = 2\varepsilon - 2z \geq 0$  and  $\phi_{zz} = -2$ . Using expression (1.16) to compute the Laplacian, we have

$$\Delta\phi = \phi_{zz} - \frac{2(m-1)}{y^2 - z^2}z\phi_z \quad \text{in } A.$$

Hence,

$$\Delta\phi + \tilde{c}\phi \leq \phi_{zz} + \tilde{c}\phi \leq -2 + 6\varepsilon^2\|\tilde{c}_+\|_{L^\infty(H)} < 0 \quad \text{in } A.$$

This finishes the proof.  $\square$

### 3. UNIQUENESS OF SADDLE-SHAPED SOLUTION

In this section we prove our uniqueness result, Theorem 1.2. We use the maximum principle of the previous section and also the following simple result.

**Lemma 3.1.** *Assume that  $f$  satisfies (1.2) and that  $u_1$  and  $u_2$  are two saddle-shaped solutions of (1.1), where  $2m \geq 2$ . Then, there exists a saddle-shaped solution  $u$  of (1.1) such that*

$$u \leq u_1 \quad \text{and} \quad u \leq u_2 \quad \text{in } \mathcal{O} = \{s > t\}. \quad (3.1)$$

This result follows from a more general one: Proposition 3.8 of [10] on the existence of a minimal saddle-shaped solution, i.e., smaller than or equal to any other saddle-shaped solution in  $\mathcal{O}$ . However, the statement of Lemma 3.1 suffices for our purposes here and, for completeness, we give next a simple proof of it.

*Proof of Lemma 3.1.* Let

$$w := \min\{u_1, u_2\} \quad \text{in } \overline{\mathcal{O}},$$

an  $H^1$  function locally in  $\overline{\mathcal{O}}$  and positive in  $\mathcal{O}$ .

For  $R > 0$ , consider the problem

$$\begin{cases} -\Delta u_R = f(u_R) & \text{in } \mathcal{O} \cap B_R(0) \\ u_R = w & \text{on } \partial(\mathcal{O} \cap B_R(0)). \end{cases} \quad (3.2)$$

By its definition,  $w$  is a weak supersolution of (3.2), while 0 is clearly a subsolution. As a consequence, there exists a weak solution  $u_R$  of (3.2) with  $0 \leq u_R \leq w$ . It can be taken to be a minimizer of the energy functional  $\mathcal{E}(\cdot, \mathcal{O} \cap B_R(0))$ , defined by (1.4), in the convex set

$$K_w := \left\{ v \in H^1(\mathcal{O} \cap B_R(0)) : v = v(s, t) \text{ a.e.,} \right. \\ \left. 0 \leq v \leq w \text{ in } \mathcal{O} \cap B_R(0), \text{ and } v \equiv w \text{ on } \partial(\mathcal{O} \cap B_R(0)) \right\}$$

of functions of  $s$  and  $t$  only. Note that  $K_w$  is weakly closed in  $H^1(\mathcal{O} \cap B_R(0))$ . For more details, see the proofs of Theorem 1.3 in [9] and of Theorem 2.4 in [17]. The set  $\mathcal{O} \cap B_R(0)$  not being Lipschitz at the origin (when  $2m \geq 4$ ) may be avoided removing from it a small ball  $B_\varepsilon(0)$ , minimizing here, and then letting  $\varepsilon \rightarrow 0$ .

Since 0 is not a weak solution of (3.2), the strong maximum principle leads to

$$0 < u_R = u_R(s, t) \leq w = w(s, t) \quad \text{in } \mathcal{O} \cap B_R(0).$$

Next, by elliptic estimates and the Arzela-Ascoli theorem (see [9, 10] for more details), the limit as  $R \rightarrow \infty$  of  $u_R$  exists (up to subsequences)



in every compact set of  $\overline{\mathcal{O}}$ . We obtain a solution  $u$  of  $-\Delta u = f(u)$  in  $\mathcal{O} = \{s > t\}$  such that  $u = 0$  on  $\mathcal{C}$  and  $0 \leq u \leq w$  in  $\mathcal{O}$ . Reflecting  $u = u(s, t)$  to be odd with respect to the Simons cone, we obtain a solution  $u = u(s, t)$  of (1.1) in all of  $\mathbb{R}^{2m}$  satisfying (3.1).

To finish the proof it remains to show that  $u > 0$  in  $\mathcal{O}$ . This will ensure that  $u$  is a saddle-shaped solution. We use the argument in (1.14); it gives that  $u_R > 0$  is a positive supersolution of the linearized operator  $\Delta + f'(u_R)$  in  $\mathcal{O} \cap B_R(0)$ . As a consequence (see section 2) the maximum principle holds for this operator in compact subdomains of  $\mathcal{O} \cap B_R(0)$ , and hence its first Dirichlet eigenvalue in these domains is positive. We deduce, by Rayleigh criterion, that  $Q_{u_R}(\xi) \geq 0$  for every smooth function  $\xi$  with compact support in  $\mathcal{O} \cap B_R(0)$  —recall that  $Q_{u_R}$  is defined in (1.5). The conclusion  $Q_{u_R}(\xi) \geq 0$  could also be verified in a different, very simple way. Simply use that  $u_R$  is a positive supersolution of the linearized operator and the integration by parts argument preceding (5.3) in section 5.

Now, letting  $R \rightarrow \infty$ , we are led to  $Q_u(\xi) \geq 0$  for all smooth functions  $\xi$  with compact support in  $\mathcal{O}$ . This would be a contradiction with  $u \equiv 0$  in  $\mathcal{O}$ , since in such case  $f'(u) = f'(0)$  is a positive constant and hence  $-\Delta - f'(0)$  is not a nonnegative operator in balls of  $\mathcal{O}$  with sufficiently large radius.

Therefore,  $u \geq 0$  and  $u \not\equiv 0$  in  $\mathcal{O}$ . It follows that  $u > 0$  in  $\mathcal{O}$ , by the strong maximum principle.  $\square$

The existence of the solution  $u_R$  in the above proof could also be shown by the monotone iteration procedure; see [10]. On the other hand, the fact that  $u > 0$  in  $\mathcal{O}$  could also be proved placing an explicit subsolution below all  $u_R$ ; see Remark 3.6 in [10].

We finish this section proving our uniqueness result.

*Proof of Theorem 1.2.* Let  $u_1$  and  $u_2$  be two saddle-shaped solutions of (1.1). Let  $u$  be the saddle-shaped solution of Lemma 3.1. Consider the difference  $v := u_i - u$  for  $i = 1$  and  $i = 2$ . We have that

$$-\Delta(u_i - u) = f(u_i) - f(u) \leq f'(u)(u_i - u) \quad \text{in } \mathcal{O} = \{s > t\},$$

since in this set  $u \leq u_i$  and  $f$  is concave in  $(0, 1)$ . Thus,

$$L_u(u_i - u) := \{\Delta + f'(u(x))\}(u_i - u) \geq 0 \quad \text{in } \mathcal{O} = \{s > t\}.$$

In addition, we have that  $u_i - u \equiv 0$  on  $\mathcal{C} = \partial\mathcal{O}$  and

$$\limsup_{x \in \mathcal{O}, |x| \rightarrow \infty} (u_i - u)(x) = 0$$

by the asymptotic result (1.6) applied to both  $u_i$  and  $u$ .

To the saddle-shaped solution  $u$ , we apply the maximum principle of Proposition 1.3 —a particular case of Proposition 2.2 proven in the previous section. We obtain that the maximum principle holds for  $L_u = \Delta + f'(u(x))$  in  $\mathcal{O}$ . Since  $v = u_i - u$  satisfies hypotheses (1.7) by the above facts, we deduce  $u_i - u \leq 0$  in  $\mathcal{O}$ . Thus, by (3.1),  $u_i - u \equiv 0$  in  $\mathcal{O}$ . Since this holds for both  $i = 1$  and  $i = 2$ , we deduce  $u_1 \equiv u_2$ , that is, uniqueness.  $\square$

#### 4. MONOTONICITY AND CONVEXITY PROPERTIES

We start this section with some regularity issues needed in the subsequent. Recall that we assume that  $f \in C^{2,\alpha}$  for some  $\alpha \in (0, 1)$ . Let  $u = u(x)$  be a bounded solution of (1.1). Since  $f(u) \in L^\infty$ , it is also an  $L^p$  function for all  $1 < p < \infty$  in every ball of radius 2, with a uniform bound on its  $L^p$ -norm in such balls. Thus,  $u \in W^{2,p} \subset C^{1,\alpha}$  (if  $p$  is taken large enough) with uniform bounds in every ball of radius 1 (i.e., with half the radius of the previous ones). Now, we have  $-\Delta u_{x_i} = f'(u)u_{x_i} \in C^\alpha$  for all indexes  $i$ , and hence  $u_{x_i} \in C^{2,\alpha}$ . But now we know  $-\Delta u_{x_i} = f'(u)u_{x_i} \in C^{1,\alpha}$ , and thus  $u_{x_i} \in C^{3,\alpha}$ . That is, we have

$$u \in C^{4,\alpha}(\mathbb{R}^{2m}) \quad \text{and} \quad D^k u \in L^\infty(\mathbb{R}^{2m}) \quad \text{if } 0 \leq |k| \leq 4. \quad (4.1)$$

Assume now that  $u = u(x) = u(s, t)$  is a bounded solution that depends only on  $s$  and  $t$  —as in the case of saddle-shaped solutions. For  $\tilde{s} \in \mathbb{R}$  and  $\tilde{t} \in \mathbb{R}$ , let

$$\tilde{u}(\tilde{s}, \tilde{t}) := u(\tilde{s}, x_2 = 0, \dots, x_m = 0, \tilde{t}, x_{m+2} = 0, \dots, x_{2m} = 0).$$

Since  $u \in C^4(\mathbb{R}^{2m})$ , we deduce that  $\tilde{u} \in C^4(\mathbb{R}^2)$  and hence  $u = u(s, t)$  is also a  $C^4$  function of the variables  $s \geq 0$  and  $t \geq 0$ . Furthermore,  $u = u(s, t)$  is the restriction to  $(s, t) \in [0, \infty) \times [0, \infty)$  of a  $C^4(\mathbb{R}^2)$  function  $\tilde{u}$  which is even in  $s$  and in  $t$ . In particular we have

$$u_s = 0 \quad \text{in } \{s = 0\} \quad \text{and} \quad u_t = 0 \quad \text{in } \{t = 0\}. \quad (4.2)$$

As a consequence,

$$u_{st} \in C^2(\mathbb{R}^{2m}) \quad \text{and} \quad u_{st} = 0 \quad \text{in } \{st = 0\}. \quad (4.3)$$

To establish the statement  $u_{st} > 0$  in  $\{s > t > 0\}$  of Proposition 1.5, we need the following asymptotic result.

**Lemma 4.1.** *Assume that  $f$  satisfies (1.2). Let  $u$  be the saddle-shaped solution of  $-\Delta u = f(u)$  in  $\mathbb{R}^{2m}$ , where  $2m \geq 2$ .*

*Then,*

$$\|D_{(s,t)}^2(u - U)\|_{L^\infty(\{st > 0, s^2 + t^2 \geq R^2\})} \longrightarrow 0 \quad \text{as } R \rightarrow \infty, \quad (4.4)$$

where  $U$  is defined in (1.3).

Recall that  $U$  is a Lipschitz function in all of  $\mathbb{R}^{2m}$ , but it is not  $C^1$  at  $\{st = 0\}$ . It is therefore important to take the sup-norm of  $D_{(s,t)}^2(u - U)$  as a function of the two variables  $s$  and  $t$ , in  $\{st > 0, s^2 + t^2 \geq R^2\}$  —which does not contain  $\{st = 0\}$ .

*Proof of Lemma 4.1.* We follow the proof of Theorem 1.6 of [10]. It argues by contradicting (1.6) —here by contradicting (4.4)— and in this way obtaining a sequence of points  $\{x_k\}$ , with  $|x_k| \rightarrow \infty$ , for which one of these asymptotics does not hold. By odd symmetry, and taking a subsequence, one may assume that  $\{x_k\} \subset \mathcal{O}$ .

Next, one translates the solution  $u$  to be centered now at  $x_k$ , and uses a translation and compactness argument; compactness comes from a priori estimates and the Arzela-Ascoli theorem. The translated solutions converge to a solution  $v$  in all of  $\mathbb{R}^{2m}$  in the  $C^4$  uniform convergence in compact sets, since any uniformly bounded sequence of solutions is uniformly bounded in  $C^4$  on every compact set, as shown above. The points  $x_k$  in the proof satisfy  $|x_k| \rightarrow \infty$  and now, in addition,  $s_k t_k > 0$  —since we are contradicting the  $L^\infty(\{st > 0, s^2 + t^2 \geq R^2\})$  convergence.

Now, in case 1 of the proof we have that the distances to the Simons cone  $|z_k| = z_k = (s_k - t_k)/\sqrt{2} \rightarrow \infty$  and thus the limiting solution  $v$  is defined in all  $\mathbb{R}^{2m}$  and satisfies  $0 \leq v \leq 1$ . By stability of  $u$  in  $\mathcal{O}$  we deduce the stability of  $v$  in  $\mathbb{R}^{2m}$ . Thus,  $v \not\equiv 0$  and therefore a Liouville-type theorem of Aronson and Weinberger [3] (see also [4] for a more general version, and [10] for the statements) guarantees that  $v \equiv 1$ . Thus  $\|D_{(s,t)}^2 u(s_k, t_k)\| \rightarrow 0$ , and since  $\|D_{(s,t)}^2 U(s_k, t_k)\| = |u_0''(z_k)| \rightarrow 0$  because  $z_k \rightarrow +\infty$ , the proof arrives at a contradiction.

Finally, in case 2 of the proof, the points  $x_k$  remain at a finite distance of the Simons cone. Since the curvatures of a cone tend to zero at infinity, in this case the limiting solution  $v$  is nonnegative in a certain limiting half-space  $\mathbb{R}_+^{2m}$  and  $v$  vanishes at its boundary. By stability again,  $v \not\equiv 0$  and hence  $v > 0$  in the half-space. Then, a Liouville theorem of Angenent [2] (see also [10] for the statement) gives that  $v$  is the 1D solution  $u_0$  depending only on the Euclidean variable orthogonal to the boundary of the half-space. Since  $\{z_k\}$  are the distances to the cone and remain bounded, in the limit this solution agrees with  $u_0(z) = U(x)$ . Hence, the full Hessian  $D_x^2(u - U)(x_k)$  tends to zero.  $\square$

We can now give the

*Proof of Proposition 1.5.* Let  $u$  be the saddle-shaped solution of (1.1). Differentiating (1.15) with respect to  $s$  and  $t$  we get

$$\Delta u_s + f'(u)u_s - \frac{m-1}{s^2}u_s = 0 \quad \text{in } \mathbb{R}^{2m} \setminus \{s=0\} \quad (4.5)$$

and

$$\Delta u_t + f'(u)u_t - \frac{m-1}{t^2}u_t = 0 \quad \text{in } \mathbb{R}^{2m} \setminus \{t=0\}. \quad (4.6)$$

Taking into account (4.5), we apply the maximum principle of Proposition 2.2 to the function  $u_s$  in  $\Omega := \{s > t\} = \mathcal{O} \subset \mathbb{R}^{2m}$  with  $c(x) := -(m-1)s^{-2}$ , a negative continuous function in  $\{s > t\}$ . Recall that  $u_s$  is  $C^2$  in all  $\mathbb{R}^{2m}$  and note that it satisfies  $u_s \geq 0$  on  $\partial\mathcal{O} = \{s=t\}$  since  $u \equiv 0$  on  $\{s=t\}$  and  $u > 0$  in  $\{s > t\}$ . Furthermore, we have  $\limsup_{x \in \mathcal{O}, |x| \rightarrow \infty} u_s(x) \geq 0$ , by the asymptotic result (1.6) and since  $U_s(x) = u'_0((s-t)/\sqrt{2})/\sqrt{2} \geq 0$ . We deduce that

$$u_s \geq 0 \quad \text{in } \mathcal{O} = \{s > t\}. \quad (4.7)$$

Next, we apply Proposition 2.2 in a different subdomain of  $\mathcal{O}$ . We apply it to the equation (4.6) and the function  $u_t$  in  $\Omega := \{s > t > 0\} \subset \mathbb{R}^{2m}$ , with  $c(x) := -(m-1)t^{-2}$ , a negative continuous function in  $\{s > t > 0\}$ . Note that  $u_t \leq 0$  on  $\partial\{s > t > 0\} = \{s=t\} \cup \{t=0\}$ ; here we use (4.2). We also have  $\limsup_{x \in \{s > t > 0\}, |x| \rightarrow \infty} u_t(x) \leq 0$ , by the asymptotic result (1.6) and since  $U_t(x) = -u'_0((s-t)/\sqrt{2})/\sqrt{2} \leq 0$ . We deduce that

$$u_t \leq 0 \quad \text{in } \{s > t > 0\}. \quad (4.8)$$

Since  $u(s, t) = -u(t, s)$  in all  $\mathbb{R}^{2m}$ , (4.7) and (4.8) lead to  $u_s \geq 0$  in all  $\mathbb{R}^{2m}$ . This, the strong maximum principle, and equation (4.5) finally give

$$u_s > 0 \quad \text{in } \mathbb{R}^{2m} \setminus \{s=0\}. \quad (4.9)$$

Symmetrically, we have

$$-u_t > 0 \quad \text{in } \mathbb{R}^{2m} \setminus \{t=0\}. \quad (4.10)$$

In particular, statement (1.12) of the proposition is now proved.

To establish (1.11), using  $\partial_y = (\partial_s + \partial_t)/\sqrt{2}$ , we obtain

$$\begin{aligned} \Delta u_y + f'(u)u_y &= \frac{m-1}{\sqrt{2}} \left( \frac{u_s}{s^2} + \frac{u_t}{t^2} \right) \\ &= \frac{m-1}{s^2}u_y + \frac{(m-1)(s^2 - t^2)}{\sqrt{2}s^2t^2}u_t \quad \text{in } \mathbb{R}^{2m} \setminus \{st=0\}. \end{aligned}$$

Thus, by (4.10), we deduce

$$\Delta u_y + f'(u)u_y - \frac{m-1}{s^2}u_y \leq 0 \quad \text{in } \mathcal{O} = \{s > t\}. \quad (4.11)$$

We apply Proposition 2.2 to the function  $u_y$  in  $\Omega := \mathcal{O} = \{s > t\} \subset \mathbb{R}^{2m}$  with  $c(x) := -(m-1)s^{-2}$ . Note that  $u_y \equiv 0$  on  $\partial\mathcal{O} = \{s = t\} = \{z = 0\}$ . Furthermore, we have  $\limsup_{x \in \mathcal{O}, |x| \rightarrow \infty} u_y(x) = 0$ , by the asymptotic result (1.6) and since  $U_y \equiv 0$  in all  $\mathbb{R}^{2m}$ . We deduce that  $u_y \geq 0$  in  $\mathcal{O} = \{s > t\}$ . This, the strong maximum principle, and (4.11) give  $u_y > 0$  in  $\mathcal{O} = \{s > t\}$ , i.e., (1.11) of the proposition.

It remains to establish (1.13). Differentiating (4.5) with respect to  $t$  and recalling the expression of the Laplacian in  $(s, t)$  variables, we obtain

$$\begin{aligned} \Delta u_{st} + f'(u)u_{st} - (m-1) \left( \frac{1}{s^2} + \frac{1}{t^2} \right) u_{st} \\ = -f''(u)u_s u_t \\ \leq 0 \quad \text{in } \{s > t > 0\}. \end{aligned} \quad (4.12)$$

We apply Proposition 2.2 to this inequality and to the  $C^2(\mathbb{R}^{2m})$  function  $u_{st}$ —recall (4.3)—in the domain  $\Omega := \{s > t > 0\} \subset \mathbb{R}^{2m}$ , with  $c(x) := -(m-1)(s^{-2} + t^{-2})$ , a negative continuous function in  $\{s > t > 0\}$ . Note that  $\partial\{s > t > 0\} = \{s = t\} \cup \{t = 0\}$  and that  $u_{st} = 0$  on  $\{t = 0\}$  by (4.3). In addition, since  $u = 0$  on  $\{s = t\} = \{z = 0\}$  we have  $u_{yy} = 0$  on  $\{s = t\} = \{z = 0\}$ . Since  $u$  is odd with respect to  $z$ , we also have  $u_{zz} = 0$  on  $\{s = t\} = \{z = 0\}$ . Thus, since

$$u_{st} = \frac{1}{2}(u_{yy} - u_{zz}),$$

we deduce that  $u_{st} = 0$  on  $\{s = t\}$ . Finally, note that

$$\limsup_{x \in \{s > t > 0\}, |x| \rightarrow \infty} u_{st}(x) \geq 0,$$

by the asymptotic result (4.4) and since  $U_{st}(x) = (1/2)(U_{yy} - U_{zz})(z) = -U_{zz}(z)/2 = -u_0''(z)/2 = f(u_0(z))/2 \geq 0$  in  $\{s > t\} = \{z > 0\}$ . Proposition 2.2 leads to  $u_{st} \geq 0$  in  $\{s > t > 0\}$ . From this, (4.12), and the strong maximum principle, we conclude the strict sign for  $u_{st}$  in  $\{s > t > 0\}$ , as stated in (1.13).  $\square$

## 5. THE SUPERSOLUTION OF THE LINEARIZED EQUATION

We end up establishing our stability result.

*Proof of Theorem 1.4.* Let  $u$  be the saddle-shaped solution of (1.1) in  $\mathbb{R}^{2m}$ . Recall that since  $2m \geq 14$ , we can take  $b > 0$  satisfying (1.8), or equivalently (1.9). Let

$$\varphi := t^{-b}u_s - s^{-b}u_t,$$

a  $C^2$  function in  $\{st > 0\}$ . By (4.9) and (4.10), we have that

$$\varphi > 0 \quad \text{in } \{st > 0\}. \quad (5.1)$$

Now, since  $u(t, s) = -u(s, t)$ , one easily verifies that  $\varphi(t, s) = \varphi(s, t)$ , i.e.,  $\varphi$  is even with respect to  $z$ . Thus  $\{\Delta + f'(u)\}\varphi$  is also even with respect to  $z$ , and hence we only need to show that  $\{\Delta + f'(u)\}\varphi \leq 0$  in  $\{s > t > 0\}$ . From this we will deduce the same inequality in all  $\{st > 0\}$  —as stated in the theorem. Then, at the end of the proof, we will show that this easily leads to the stability of  $u$  in all of  $\mathbb{R}^{2m}$ .

In  $\{s > t > 0\}$ , we have

$$\Delta t^{-b} = b(b - m + 2)t^{-b-2} \quad \text{and} \quad \Delta s^{-b} = b(b - m + 2)s^{-b-2}.$$

Thus, using also (4.5) and (4.6), in  $\{s > t > 0\}$

$$\begin{aligned} \Delta \varphi &= b(b - m + 2)t^{-b-2}u_s \\ &\quad - f'(u)u_s t^{-b} + (m - 1)s^{-2}u_s t^{-b} - 2bt^{-b-1}u_{st} \\ &\quad - b(b - m + 2)s^{-b-2}u_t \\ &\quad - \{-f'(u)u_t s^{-b} + (m - 1)t^{-2}u_t s^{-b} - 2bs^{-b-1}u_{st}\}, \end{aligned}$$

and hence

$$\begin{aligned} \{\Delta + f'(u)\}\varphi &= t^{-b}u_s\{(m - 1)s^{-2} + b(b - m + 2)t^{-2}\} \\ &\quad - s^{-b}u_t\{(m - 1)t^{-2} + b(b - m + 2)s^{-2}\} \\ &\quad + 2bu_{st}\{s^{-b-1} - t^{-b-1}\}. \end{aligned}$$

Now, using that, in  $\{s > t > 0\}$ ,  $u_{st} > 0$ ,  $u_y > 0$ , and  $-u_t > 0$  (by Proposition 1.5), and also the inequality (1.8) for  $b > 0$ , we arrive at

$$\begin{aligned} \{\Delta + f'(u)\}\varphi &\leq t^{-b}(u_s + u_t)\{(m - 1)s^{-2} + b(b - m + 2)t^{-2}\} \\ &\quad - s^{-b}u_t\{(m - 1)t^{-2} + b(b - m + 2)s^{-2}\} \\ &\quad - t^{-b}u_t\{(m - 1)s^{-2} + b(b - m + 2)t^{-2}\} \\ &= u_y \sqrt{2}t^{-b}\{(m - 1)s^{-2} + b(b - m + 2)t^{-2}\} \\ &\quad + (-u_t)(m - 1)(s^{-b}t^{-2} + t^{-b}s^{-2}) \\ &\quad + (-u_t)b(b - m + 2)(s^{-2-b} + t^{-2-b}) \\ &\leq u_y \sqrt{2}t^{-b}(m - 1)\{s^{-2} - t^{-2}\} \\ &\quad + (-u_t)(m - 1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b}) \\ &\leq (-u_t)(m - 1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b}) \end{aligned}$$

in  $\{s > t > 0\}$ . Finally, since in  $\{s > t > 0\}$  we have  $-u_t > 0$  and

$$\begin{aligned} s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b} &= s^{-b}(t^{-2} - s^{-2}) + t^{-b}(s^{-2} - t^{-2}) \\ &= (s^{-b} - t^{-b})(t^{-2} - s^{-2}) \leq 0, \end{aligned}$$

we conclude  $\{\Delta + f'(u)\}\varphi \leq 0$  in  $\{s > t > 0\}$ . Hence, by even symmetry in  $z$ , also

$$\{\Delta + f'(u)\}\varphi \leq 0 \quad \text{in } \mathbb{R}^{2m} \setminus \{st = 0\} = \{st > 0\}. \quad (5.2)$$

Next, using (5.1) and (5.2), we can verify the stability condition for any  $C^1$  test function  $\xi = \xi(x)$  with compact support in  $\{st > 0\}$ . Indeed, multiply (5.2) by  $\xi^2/\varphi$  and integrate by parts to get

$$\begin{aligned} \int_{\{st>0\}} f'(u) \xi^2 dx &= \int_{\{st>0\}} f'(u) \varphi \frac{\xi^2}{\varphi} dx \\ &\leq \int_{\{st>0\}} -\Delta \varphi \frac{\xi^2}{\varphi} dx \\ &= \int_{\{st>0\}} \nabla \varphi \nabla \xi \frac{2\xi}{\varphi} dx - \int_{\{st>0\}} \frac{|\nabla \varphi|^2}{\varphi^2} \xi^2 dx. \end{aligned}$$

Now, using the Cauchy-Schwarz inequality, we are led to

$$\int_{\{st>0\}} f'(u) \xi^2 dx \leq \int_{\{st>0\}} |\nabla \xi|^2 dx. \quad (5.3)$$

Finally, we need to prove this same inequality for every  $C^1$  function  $\xi$  with compact support in a ball  $B_{R_0}(0) \subset \mathbb{R}^{2m}$ . For this, let  $\eta_\varepsilon$  be a smooth function in  $[0, \infty)$  with  $0 \leq \eta \leq 1$ , being identically 0 in  $[0, \varepsilon/2)$  and identically 1 in  $[\varepsilon, \infty)$ . Since  $\xi(x)\eta_\varepsilon(s)\eta_\varepsilon(t)$  is a  $C^1$  function of  $x$  with compact support in  $\{s \geq \varepsilon/2, t \geq \varepsilon/2\}$ , the stability property just proven gives

$$\int_{\mathbb{R}^{2m}} f'(u(x)) \xi^2(x) \eta_\varepsilon^2(s) \eta_\varepsilon^2(t) dx \leq \int_{B_{R_0}(0)} |\nabla_x \{\xi(x) \eta_\varepsilon(s) \eta_\varepsilon(t)\}|^2 dx.$$

We now compute all the terms in the right hand side of this inequality and, using Cauchy-Schwarz, we see that to conclude

$$\int_{\mathbb{R}^{2m}} f'(u) \xi^2 dx \leq \int_{\mathbb{R}^{2m}} |\nabla \xi|^2 dx$$

by letting  $\varepsilon \rightarrow 0$ , it is enough to use that

$$\begin{aligned} \int_{B_{R_0}(0)} |\nabla_x \eta_\varepsilon(s)|^2 dx &\leq \int_{\{s \leq \varepsilon, t \leq R_0\}} |\nabla_x \eta_\varepsilon(s)|^2 dx \\ &\leq \int_{\{s \leq \varepsilon, t \leq R_0\}} C \varepsilon^{-2} s^{m-1} t^{m-1} ds dt \\ &\leq C \varepsilon^{m-2} R_0^m \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \end{aligned}$$

since  $m \geq 3$  —and the same for the integral of  $|\nabla_x \eta_\varepsilon(t)|^2$ . This concludes the proof.  $\square$

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ICREA AND UNIVERSITAT POLITÈCNICA DE CATALUNYA, DEPARTAMENT DE  
MATEMÀTICA APLICADA I, DIAGONAL 647, 08028 BARCELONA, SPAIN  
*E-mail address:* `xavier.cabre@upc.edu`